§. Properties of A_p weights:

Let $w \in A_p$ for $1<p<\infty$. Then:

(1). $[\lambda w]_{A_p} = [w]_{A_p}$, $\forall \lambda>0$

(2). The weight $\boxed{w' := w^{-\frac{1}{p-1}}}$ belongs to the class $A_{p'}$ with characteristic constant:

$$\boxed{[w']_{A_{p'}} = [w]_{A_p}^{p'-1}}$$

(3). $[w]_{A_p} \geq 1$, $\forall w \in A_p$. Equality holds if & only if $w = $ constant.

(4). The A_p classes are increasing:

$$[w]_{A_q} \leq [w]_{A_p}, \quad \forall 1<p<q<\infty.$$

(5). Equivalent characterization of the A_p characteristic:

$$\boxed{[w]_{A_p} = \sup_{Q \text{ cube}} \; \sup_{\substack{f \in L^p(Q,w) \\ |\{Q \cap \{|f|=0\}\}|=0}} \frac{\langle |f| \rangle_Q^p}{\mathbb{E}_Q^w(|f|^p)}} \qquad \frac{\left(\frac{1}{|Q|}\int_Q |f(x)|dx\right)^p}{\frac{1}{w(Q)}\int_Q |f|^p dw}$$

(6). A_p measures are doubling measures:

$$\boxed{w(\lambda Q) \leq \lambda^{np} [w]_{A_p} w(Q)}$$

(λQ = the cube w/ same center as Q, and side length $\lambda \cdot \ell(Q)$).

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(1). $[\lambda w]_{A_p} = \sup_Q \langle \lambda w \rangle_Q \left\langle \frac{1}{\lambda^{p-1} w^{p-1}} \right\rangle_Q^{p-1} = \sup_Q \langle w \rangle_Q \langle w^{-\frac{1}{p-1}} \rangle_Q^{p-1} = [w]_{A_p}$. $\qquad \frac{p'}{p} = \frac{1}{p-1}$

(2). Previous notes

(3). $1 = \frac{1}{|Q|}\int_Q 1 dx = \frac{1}{|Q|}\int_Q w^{\frac{1}{p}} w^{-\frac{1}{p}} \leq \frac{1}{|Q|}\left(\int_Q w\right)^{1/p}\left(\int_Q \overbrace{w^{-p'/p}}^{w'}\right)^{1/p'} = \langle w \rangle_Q^{1/p} \langle w' \rangle_Q^{1/p'}$

$\Rightarrow 1 \leq \langle w \rangle_Q \langle w' \rangle_Q^{p/p' = p-1} \Rightarrow 1 \leq \sup_Q \langle w \rangle_Q \langle w' \rangle_Q^{p-1} = [w]_{A_p}$.

If $[w]_{A_p} = 1$, then $\langle w \rangle_Q \langle w' \rangle_Q^{p-1} = 1, \forall Q \Rightarrow$ equality in Hölder above $\Rightarrow w^{1/p} = c \cdot w^{-1/p} \Rightarrow w \equiv c$.

(4). Let $1<p<q<\infty$, and suppose $w \in A_p$.

$[w]_{A_q} = \sup_Q \langle w \rangle_Q \langle w^{-\frac{1}{q-1}} \rangle_Q^{q-1}$

$p<q \Rightarrow p_1 := \frac{q-1}{p-1} > 1 \Rightarrow$ can do Hölder with p_1 and $p_1' = \frac{p_1}{p_1 - 1} = \frac{q-1}{q-p}$

$\langle w^{-\frac{1}{q-1}} \rangle_Q = \frac{1}{|Q|}\int_Q w^{-\frac{1}{q-1}} \leq \frac{1}{|Q|}\left(\int_Q w^{-\frac{1}{q-1}\frac{q-1}{p-1}}\right)^{1/p_1} |Q|^{1/p_1'} = \langle w^{-\frac{1}{p-1}} \rangle_Q^{1/p_1} = \langle w^{-\frac{1}{p-1}} \rangle_Q^{\frac{p-1}{q-1}}$

$\Rightarrow [w]_{A_q} = \sup_Q \langle w \rangle_Q \langle w^{-\frac{1}{q-1}} \rangle_Q^{q-1} \leq \sup_Q \langle w \rangle_Q \langle w^{-\frac{1}{p-1}} \rangle_Q^{p-1} = [w]_{A_p}$.

(5). $$\langle |f| \rangle_Q = \frac{1}{|Q|}\int_Q |f| = \frac{1}{|Q|}\int_Q |f|\, w^{+1/p} w^{-1/p} \le \frac{1}{|Q|}\left(\int_Q |f|^p dw\right)^{1/p}\left(\int_Q w^{-p'/p}\right)^{1/p'}$$

$$= \frac{1}{|Q|}\left(\int_Q |f|^p dw\right)^{1/p} w'(Q)^{1/p'} \qquad \langle w'\rangle_Q^{p-1 = \frac{p}{p'}} \le [w]_{A_p}\frac{1}{\langle w\rangle_Q}$$

$$= \left(\frac{1}{|Q|}\int_Q |f|^p dw\right)^{1/p}\langle w'\rangle_Q^{1/p'} \le \left(\frac{1}{|Q|}\int_Q |f|^p dw\right)^{1/p}[w]_{A_p}^{1/p}\frac{1}{\langle w\rangle_Q^{1/p}}$$

$$= \cancel{\frac{1}{|Q|^{1/p}}}\left(\int_Q |f|^p dw\right)^{1/p}[w]_{A_p}^{1/p}\frac{\cancel{|Q|^{1/p}}}{w(Q)^{1/p}}$$

$$= \left(\mathbb{E}_Q^w(|f|^p)\right)^{1/p}[w]_{A_p}^{1/p}$$

$\Rightarrow \langle |f|\rangle_Q^p \le \mathbb{E}_Q^w(|f|^p)[w]_{A_p}$, $\forall Q$ cube

$\Rightarrow \dfrac{\langle |f|\rangle_Q^p}{\mathbb{E}_Q^w(|f|^p)} \le [w]_{A_p}$, $\forall Q$ cube $\Rightarrow \sup\limits_{Q,f}\dfrac{\langle |f|\rangle_Q^p}{\mathbb{E}_Q^w(|f|^p)} \le [w]_{A_p}$

<u>Converse</u>: $$\sup_{Q,f}\frac{\langle |f|\rangle_Q^p}{\mathbb{E}_Q^w(|f|^p)} \ge \frac{\langle w'\rangle_Q^p}{\mathbb{E}_Q^w((w')^p)} = \frac{w'(Q)^p}{|Q|^p}\,\frac{w(Q)}{w'(Q)} = \langle w\rangle_Q\langle w'\rangle_Q^{p-1}, \ \forall Q$$

$$\Downarrow$$

$$[w]_{A_p} \le \sup_{Q,f}\frac{\langle |f|\rangle_Q^p}{\mathbb{E}_Q^w(|f|^p)}$$

$$\mathbb{E}_Q^w((w')^p) = \mathbb{E}_Q^w(w^{-p'}) = \frac{1}{w(Q)}\int_Q w^{1-p'} = \frac{w'(Q)}{w(Q)}$$

<u>Remark</u>: Where does the [$f \neq 0$ a.e. on Q] condition come in?
When taking $f = w'$, and needing $\langle w'\rangle_Q$, or $\int_Q w^{-p'/p} = \int_Q \frac{1}{w^{p'/p}} < \infty$,
we need w to only be 0 on a null subset of Q.

(6). Let $w \in A_p$ and let Q_0 be a cube in $\mathbb{R}^n$. In:

$$\sup_{Q,f}\frac{\langle |f|\rangle_Q^p}{\mathbb{E}_Q^w(|f|^p)} \le [w]_{A_p} \quad \text{take } f = \mathbb{1}_{Q_0}: \ \sup_Q\frac{\langle \mathbb{1}_{Q_0}\rangle_Q^p}{\mathbb{E}_Q^w(\mathbb{1}_{Q_0})} = \sup_Q\left(\frac{w(Q)}{w(Q_0)}\,\frac{|Q\cap Q_0|^p}{|Q|^p}\right) \le [w]_{A_p}$$

$$\frac{\langle \mathbb{1}_{Q_0}\rangle_Q^p}{\mathbb{E}_Q^w(\mathbb{1}_{Q_0})} = \frac{\left(\frac{|Q\cap Q_0|}{|Q|}\right)^p}{\frac{w(Q_0)}{w(Q)}}$$

Take $Q = \lambda Q_0$:

$$[w]_{A_p} \ge \frac{w(\lambda Q_0)}{w(Q_0)}\,\frac{|Q_0|^p}{|\lambda Q_0|^p} = \frac{w(\lambda Q_0)}{w(Q_0)}\,\frac{1}{\lambda^{np}}$$

$$\Rightarrow \boxed{w(\lambda Q_0) \le [w]_{A_p}\lambda^{np}\, w(\lambda Q_0),} \quad \forall Q_0 \text{ cube}$$

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Remark: What about the implication $\boxed{w \in A_p} \Rightarrow \boxed{M : L^p(w) \to L^p(w) \text{ bounded}}$?

Focus for a moment on the dyadic maximal function $M_{\mathcal{D}}$ and dyadic A_p weights.

Recall that we already showed the weighted dyadic maximal function

$$M_{\mathcal{D}}^{w} f(x) := \sup_{\substack{Q \in \mathcal{D} \\ x \in Q}} \mathbb{E}_Q^w |f|$$

is $L^1(w) \to L^{1,\infty}(w)$ bounded:

$$w\{M_{\mathcal{D}}^w f(x) > \lambda\} \leq \frac{1}{\lambda} \|f\|_{L^1(w)}$$

Boundedness $T : L^p(\mu) \to L^{p,\infty}(\mu)$:

$$\mu\{|Tf| > \lambda\} \leq \left(\frac{C}{\lambda} \|f\|_{L^p(\mu)}\right)^p$$

(C = norm)

Apply to $|f|^p$:

$$w\{ M_{\mathcal{D}}^w(|f|^p) > \lambda\} \leq \frac{1}{\lambda} \|f\|^p_{L^p(w)}$$

$$w\{(M_{\mathcal{D}}^w(|f|^p))^{1/p} > \lambda^{1/p}\} \leq \frac{1}{\lambda} \|f\|^p_{L^p(w)} \qquad \lambda \mapsto \lambda^p$$

$$w\{(M_{\mathcal{D}}^w(|f|^p))^{1/p} > \lambda\} \leq \frac{1}{\lambda^p} \|f\|^p_{L^p(w)}$$

$\Rightarrow$ The operator $f \mapsto (M_{\mathcal{D}}^w(|f|^p))^{1/p}$ is bounded $L^p(w) \to L^{p,\infty}(w)$! (w/ norm ≤ 1)

Now we already saw: $\langle |f| \rangle_Q^p \leq \mathbb{E}_Q^w(|f|^p)\, [w]_{A_p}$

$$\Rightarrow \quad M_{\mathcal{D}} f \leq (M_{\mathcal{D}}^w(|f|^p))^{1/p}\, [w]_{A_p}^{1/p}$$

$$\Rightarrow \quad \boxed{M_{\mathcal{D}} : L^p(w) \to L^{p,\infty}(w) \text{ is bounded!}}$$

IF ONLY! We had instead $M_{\mathcal{D}} : L^{p-\varepsilon}(w) \to L^{p-\varepsilon,\infty}(w)$!
Then combined w/ (trivial est.) $M_{\mathcal{D}} : L^\infty(w) \to L^\infty(w)$

} INTERPOLATE!

$\Downarrow$

$L^q(w) \to L^q(w)$ boundedness of $M_{\mathcal{D}}$ for all $q > p - \varepsilon$

$\Rightarrow$ FOR p!

Turns out, WE DO! We will show that

$w \in A_p \Rightarrow \exists\, \varepsilon > 0$ s.t. $w \in A_{p-\varepsilon}$!

and then the maximal function bound we want is proved!

This will be the key part to proving: $\boxed{A_p = \bigcup_{q \in (1,p)} A_q}$

Already saw: $1 < q < p < \infty \Rightarrow A_q \subset A_p$

$\Rightarrow \bigcup_{q \in (1,p)} A_q \subset A_p$. Showing the converse:

$\boxed{A_p \subset \bigcup_{q \in (1,p)} A_q}$ means exactly showing that $\forall w \in A_p$, $\exists\, q < p$ s.t. $w \in A_q$.

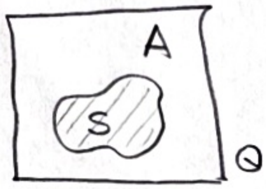
Lemma: Let $w \in A_p$ $(1<p<\infty)$ and $\alpha \in (0,1)$. Then for every cube Q and every measurable subset $S \subset Q$ with $|S| \leq \alpha |Q|$, we have

$$\boxed{w(S) \leq \beta\, w(Q), \text{ where } \beta = 1 - \frac{(1-\alpha)^p}{[w]_{A_p}}}$$

(% of mass of S in Q wrt w is proportional to the % of mass of S in Q wrt Lebesgue measure!)

Proof: Recall that we showed:

$$\langle |f| \rangle_Q^p \leq [w]_{A_p} \mathbb{E}_Q^w(|f|^p)$$



Take $f = \mathbb{1}_A$, for some subset $A \subset Q$.

$$\langle \mathbb{1}_A \rangle_Q^p = \frac{|A|^p}{|Q|^p} \; ; \quad \mathbb{E}_Q^w(\mathbb{1}_A) = \frac{w(A)}{w(Q)} \quad \Rightarrow \quad \frac{|A|^p}{|Q|^p} \leq [w]_{A_p} \frac{w(A)}{w(Q)}$$

$$\Rightarrow \boxed{\left(\frac{|A|}{|Q|}\right)^p \leq [w]_{A_p} \frac{w(A)}{w(Q)}} \qquad \forall A \subset Q$$

Now let $A = Q \setminus S$ above:

$$\left(\frac{|Q|-|S|}{|Q|}\right)^p \leq [w]_{A_p} \frac{w(Q)-w(S)}{w(Q)}$$

$$\left(1 - \frac{|S|}{|Q|}\right)^p \leq [w]_{A_p} \left(1 - \frac{w(S)}{w(Q)}\right)$$

$$\Rightarrow 1 - \frac{w(S)}{w(Q)} \geq \frac{\left(1 - |S|/|Q|\right)^p}{[w]_{A_p}} \geq \frac{(1-\alpha)^p}{[w]_{A_p}} \Rightarrow \frac{w(S)}{w(Q)} \leq 1 - \frac{(1-\alpha)^p}{[w]_{A_p}} \;//.$$

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This will be instrumental in proving the Reverse Hölder Property of A_p weights

$\forall w \in A_p$, $\exists\, C, \gamma > 0$ (depending on $n, p, [w]_{A_p}$) s.t.

$\forall$ cube Q:

$$\boxed{\left(\frac{1}{|Q|}\int_Q w(x)^{1+\gamma} dx\right)^{\frac{1}{1+\gamma}} \leq \frac{C}{|Q|}\int_Q w(x)\,dx}$$